

# General energy decay for a viscoelastic equation of Kirchhoff type with acoustic boundary conditions

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## Abstract

This paper is concerned with a viscoelastic equation of Kirchhoff type with acoustic boundary conditions in a bounded domain of  $\mathbb{R}^n$ . We show that, under suitable conditions on the initial data, the solution exists globally in time. Then, we prove the general energy decay of global solutions by applying a lemma of P. Martinez, which allows us to get our decay result for a class of relaxation functions wider than that usually used.

**Keywords and phrases :** Kirchhoff type equation, Energy decay, Acoustic boundary conditions, Memory term, Multiplier method.

**AMS Classification :** 35L72, 35A01, 35B40.

## 1 Introduction

In this paper we are concerned with the decay rates of solutions for the following nonlinear wave equation of Kirchhoff type, with acoustic boundary conditions

$$\left\{ \begin{array}{ll} u_{tt} - \left( a + b \|\nabla u\|_2^{2\gamma} \right) \Delta u + \int_0^t g(t-s) \Delta u(s) ds = |u|^{k-2}u, & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ \left( a + b \|\nabla u\|_2^{2\gamma} \right) \frac{\partial u}{\partial \nu} - \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(s) ds = y_t & \text{on } \Gamma_1 \times (0, +\infty), \\ u_t + p(x) y_t + q(x) y = 0 & \text{on } \Gamma_1 \times (0, +\infty), \\ u(., 0) = u_0, u_t(., 0) = u_1, & \text{in } \Omega, \\ y(., 0) = y_0, & \text{on } \Gamma_1, \end{array} \right. \quad (1.1)$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$ ,  $n \geq 1$  having a smooth boundary  $\Gamma = \partial\Omega$ , consisting of two closed and disjoint parts  $\Gamma_0$  and  $\Gamma_1$ . Here  $\nu$  denotes the unit outward normal to  $\Gamma$ . The parameters  $a > 0$ ,  $b > 0$ ,  $\gamma \geq 0$  and  $k > 2$  are

constant real numbers,  $p$  and  $q$  are given functions satisfying some conditions to be specified later;  $u_0, u_1 : \Omega \rightarrow \mathbb{R}$  and  $y_0 : \Gamma_1 \rightarrow \mathbb{R}$  are given functions.

The acoustic boundary conditions were introduced by Beale and Rosencrans in [6], [7]; where the authors proved the global existence and regularity of solutions of the wave equation subject to boundary conditions of the form

$$\begin{cases} \frac{\partial u}{\partial \nu} = y_t & \text{on } \Gamma \times (0, +\infty), \\ \rho u_t + m(x) y_{tt} + p(x) y_t + q(x) y = 0 & \text{on } \Gamma \times (0, +\infty), \end{cases} \quad (1.2)$$

where  $\rho > 0$ ,  $p$  is a nonnegative function and  $m, q$  are strictly positive functions on the boundary.

Wave equations with acoustic boundary conditions have been treated by many authors [4], [5], [2], [8], [19], [9]. In [4], the authors studied a linear wave equation subject to the boundary conditions (1.2), with  $m \equiv 0$ , on the portion  $\Gamma_1$  of the boundary and Dirichlet boundary conditions on the portion  $\Gamma_0$ . They proved global solvability and uniform energy decay.

Boukhatem and Benabderrahmane [19], studied the variable-coefficient viscoelastic wave equation

$$\begin{cases} u_{tt} + Lu - \int_0^t g(t-s) Lu(s) ds = |u|^{k-2} u, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ \frac{\partial u}{\partial \nu_L} - \int_0^t g(t-s) \frac{\partial u}{\partial \nu_L}(s) ds = h(x) y_t, & \text{on } \Gamma_1 \times (0, \infty), \\ u_t + p(x) y_t + q(x) y = 0, & \text{on } \Gamma_1 \times (0, \infty), \end{cases} \quad (1.3)$$

where  $Lu = -\operatorname{div}(A \nabla u)$ ,  $\frac{\partial u}{\partial \nu_L} = (A \nabla u) \cdot \nu$  and  $A = (a_{ij}(x))$  a matrix with  $a_{ij} \in C^1(\overline{\Omega})$ . Combining the techniques used by Georgiev and Todorova in [17], those used by Frota and Larkin in [4], and Faedo–Galerkin’s approximations, the authors proved global solvability for suitable initial data, and general energy decay for some relaxation functions  $g$  satisfying some known conditions, introduced firstly by Messaoudi in [14].

For quasilinear equations, the authors of [3] studied the Carrier equation

$$u_{tt} - M\left(\|u\|_2^2\right) \Delta u + |u_t|^\alpha u_t = f(u), \quad \text{in } \Omega \times (0, +\infty), \quad (1.4)$$

subject to the boundary conditions (1.2) on the portion  $\Gamma_1$  and Dirichlet boundary conditions on the portion  $\Gamma_0$ , where  $M : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a  $C^1$  function satisfying some conditions. They proved the existence and uniqueness of global solutions. In [20], the author studied the following problem

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right) \Delta u + 2\delta u_t = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ M\left(\|\nabla u\|_2^2\right) \frac{\partial u}{\partial \nu} = y_t & \text{on } \Gamma_1 \times (0, +\infty), \\ u_t + p(x) y_t + q(x) y = 0 & \text{on } \Gamma_1 \times (0, +\infty), \end{cases} \quad (1.5)$$

where  $M\left(\|\nabla u\|_2^2\right) = a + b \|\nabla u\|_2^2$ , with  $a, b > 0$ . He proved the uniform stability of solutions for a sufficiently small positive passive viscous damping coefficient  $\delta$ ,

using multiplier technique.

Recently, Lee et al in [9] were concerned with the following Kirchhoff type equation with Balakrishnan-Taylor damping, time-varying delay and boundary conditions:

$$\left\{ \begin{array}{ll} |u_t|^\rho u_{tt} - (M(u)(t)) \Delta u - \Delta u_{tt} \\ \quad + \int_0^t g(t-s) \Delta u(s) ds + |u_t|^q u_t = |u|^p u & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\ (M(u)(t)) \frac{\partial u}{\partial \nu} + \frac{\partial u_{tt}}{\partial \nu} - \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(s) ds \\ \quad + \mu_0 u_t(x, t) + \mu_1 u_t(x, t - \tau(t)) = h(x) y_t & \text{on } \Gamma_1 \times (0, +\infty), \\ u_t + p(x) y_t + q(x) y = 0 & \text{on } \Gamma_1 \times (0, +\infty), \\ u_t(x, t - \tau(t)) = f_0(x, t) & \text{on } \Gamma_1, -\tau(0) \leq t \leq 0, \end{array} \right. \quad (1.6)$$

where  $M(u)(t) = a + b \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)$ , with  $a, b, \sigma > 0$ . They showed the global existence of solutions and established a general energy decay

Motivated by the previous works, we consider a non degenerate Kirchhoff type wave equation with memory term, acoustic boundary conditions and source term. This model is new because has not been considered by predecessors. Further, we do not use differential inequalities to prove our decay result, but we use a method based on integral inequalities, introduced by P. Martinez [13] (Lemma 4.1 below) and that generalize those introduced by V. Komornik [18] and A. Haraux [1]. This method allows us to consider a class of relaxation functions larger than the one usually considered.

The paper is organized as follows. In Section 2, we present some notations and material needed for our work and state an existence result, which can be proved by combining the techniques used in [10] and [19]. In section 3, we prove that, for suitable initial data, the solution exists globally in time. In this section, our proof technique follows the arguments of [16], with some modifications being needed for our problem. Section 4 contains the statement and the proof of our main result.

## 2 Preliminaries and some notations

In this section, we present some notations and some material needed in the proof of our result.

Let  $V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$  be the subspace of the classical Sobolev space  $H^1(\Omega)$  of real valued functions of order one. The Poincaré's inequality holds on  $V$ , i.e. there exists a positive constant  $C_*$  (depends only on  $\Omega$ ) such that:

$$\|u\|_k \leq C_* \|\nabla u\|_2, \quad \text{for every } u \in V, \text{ and } 1 \leq k \leq \bar{k} \quad (2.1)$$

where  $\|u\|_k^k = \int_\Omega |u|^k dx$ ,  $\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2 dx$  and  $\bar{k} = \begin{cases} \frac{2n}{n-2}, & \text{if } n \geq 3 \\ +\infty, & \text{if } n = 1 \text{ or } 2 \end{cases}$ .

According to the trace theory, there exists a positive constant  $\overline{C}_*$  (depends only on  $\Gamma_1$ ) such that:

$$\|u\|_{2,\Gamma_1} \leq \overline{C}_* \|\nabla u\|_2, \text{ for every } u \in V, \quad (2.2)$$

where  $\|u\|_{2,\Gamma_1}^2 = \int_{\Gamma_1} |u|^2 dx$ .

To prove our result, we need the following assumptions.

**(H1)** For the functions  $p$  and  $q$ , we assume that  $p, q \in C(\Gamma_1)$  and  $p(x) > 0$ ,  $q(x) > 0$ , for all  $x \in \Gamma_1$ . This assumption assures us that there exist positive constants  $p_i, q_i$  ( $i \in \{0, 1\}$ ) such that:

$$0 < p_0 \leq p(x) \leq p_1; \quad 0 < q_0 \leq q(x) \leq q_1, \text{ for all } x \in \Gamma_1. \quad (2.3)$$

**(H2)** For the relaxation function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , we assume that :

$$g(0) > 0 \quad \text{and} \quad a - \int_0^\infty g(s) ds = l > 0. \quad (2.4)$$

and there exists a locally absolutely continuous function  $\xi : [0, +\infty) \rightarrow (0, +\infty)$ , and constants  $\theta \geq 0, r$  such that

$$\int_0^\infty \xi(s) ds = +\infty, \quad g'(t) \leq -\xi(t)g(t), \text{ for all } t \geq 0, . \quad (2.5)$$

$$\frac{\xi'}{\xi^\theta} \in L^1(0, +\infty), \quad \int_t^{t+s} \frac{\xi'(\tau)}{\xi(\tau)} d\tau \leq r, \text{ for all } t, s \geq 0, . \quad (2.6)$$

The last assumption assures us that  $\xi(t+s) \leq e^r \xi(t)$ , for all  $t, s \geq 0$ , and so  $\xi \in L^\infty(0, +\infty)$ .

**Remark 2.1** (i) Notice that, if  $\xi$  is a nonincreasing function, then hypothesis (2.6) is trivially valid with  $\theta = r = 0$ .

(ii) Notice also that, proofs in previous papers depend strongly on the non-increase of  $\xi$ . We prove our result without this restriction.

We now state a local existence theorem for problem (1.1), whose proof follows the arguments in [19].

**Theorem 2.2 (Local existence)** Let  $2 < k \leq \frac{2n-2}{n-2}$ ,  $(u_0, u_1) \in V \times L^2(\Omega)$  and  $y_0 \in L^2(\Gamma_1)$ . Suppose that hypotheses (H1)–(H2) hold, then there exists a unique pair of functions  $(u, y)$ , which is a solution of the problem (1.1) such that

$$\begin{aligned} u &\in C(0, T; V), & u_t &\in C(0, T; L^2(\Omega)), \\ y &\in L^\infty(0, T; L^2(\Gamma_1)), & y_t &\in L^2(0, T; L^2(\Gamma_1)); \end{aligned}$$

for some  $T > 0$ .

### 3 Global existence

In this section, we shall state and prove the global existence. For this purpose, we define the energy of problem (1.1) by

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + J(u, y)(t), \quad (3.1)$$

where

$$\begin{aligned} J(u, y)(t) = & \frac{1}{2} \left( a - \left( \int_0^t g(s) ds \right) \right) \|\nabla u\|_2^2 + \frac{b}{\gamma+1} \|\nabla u\|_2^{2(\gamma+1)} \\ & + \int_{\Gamma_1} \frac{1}{2} (q(x) y^2) dx + \frac{1}{2} (g \diamond \nabla u)(t) - \frac{1}{k} \|u\|_k^k, \end{aligned} \quad (3.2)$$

and

$$(g \diamond \nabla u)(t) = \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|_2^2 ds. \quad (3.3)$$

**Lemma 3.1** *Let  $(u, y)$  be the solution of (1.1). Then, the energy functional defined by (3.1) is a nonincreasing function and*

$$E'(t) = -\frac{1}{2} g(t) \|\nabla u\|_2^2 + \frac{1}{2} (g' \diamond \nabla u)(t) - \int_{\Gamma_1} p(x) y_t^2 dx, \text{ for all } t \geq 0. \quad (3.4)$$

**Proof.** Multiplying the first equation in (1.1) by  $u_t$ , integrating over  $\Omega$ , using integration by parts and exploiting the third equation in system (1.1), we get

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} a \|\nabla u\|_2^2 + \frac{b}{2(\gamma+1)} \|\nabla u\|_2^{2(\gamma+1)} - \frac{1}{2} \left( \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \right. \\ \left. + \frac{1}{2} (g \diamond \nabla u)(t) - \frac{1}{k} \|u\|_k^k \right) = \int_{\Gamma_1} u_t y_t dx - \frac{1}{2} g(t) \|\nabla u\|_2^2 + \frac{1}{2} (g' \diamond \nabla u)(t) \end{aligned} \quad (3.5)$$

From the fourth equation in (1.1), we deduce that

$$\int_{\Gamma_1} u_t y_t dx = - \int_{\Gamma_1} p(x) y_t^2 dx - \frac{d}{dt} \left( \int_{\Gamma_1} \frac{1}{2} (q(x) y^2) dx \right). \quad (3.6)$$

Plugging (3.6) into (3.5) and making use of (3.1), then we obtain (3.4), and hence  $E'(t) \leq 0$ , for all  $t \geq 0$ . ■

As in [11] and [16], we define a functional  $F$ , which helps in establishing the global existence of solution.

Setting

$$F(x) = \frac{1}{2} x^2 - \frac{B_\Omega^k}{k} x^k \quad (3.7)$$

where

$$B_{\Omega} = \sup_{u \in V, u \neq 0} \frac{\|u\|_k}{\sqrt{l \|\nabla u\|_2^2 + \frac{b}{\gamma+1} \|\nabla u\|_2^{2(\gamma+1)}}} \quad (3.8)$$

We can verify, as in [11], that the function  $F$  is increasing in  $(0, \lambda_1)$  and decreasing in  $(\lambda_1, +\infty)$ , where  $\lambda_1$  is the strictly positive zero of the derivative function  $F'$ , that is

$$\lambda_1 = B_{\Omega}^{\frac{-k}{k-2}}. \quad (3.9)$$

$F$  has a maximum at  $\lambda_1$  with the maximum value

$$d_1 = F(\lambda_1) = \left( \frac{k-2}{2k} \right) B_{\Omega}^{\frac{-2k}{k-2}} \quad (3.10)$$

From (3.1), (3.2), (3.8), (2.4) and (3.7), we have

$$\begin{aligned} E(t) &\geq J(u, y)(t) = \frac{1}{2} \left( a - \left( \int_0^t g(s) ds \right) \right) \|\nabla u\|_2^2 + \frac{b}{2(\gamma+1)} \|\nabla u\|_2^{2(\gamma+1)} \\ &\quad + \int_{\Gamma_1} \frac{1}{2} (q(x) y^2) dx + \frac{1}{2} (g \diamond \nabla u)(t) - \frac{1}{k} \|u\|_k^k \\ &\geq \frac{1}{2} \gamma^2(t) - \frac{1}{k} B_{\Omega}^k \gamma^k(t) = F(\gamma(t)) \end{aligned} \quad (3.11)$$

where

$$\gamma(t) = \sqrt{l \|\nabla u\|_2^2 + \frac{b}{(\gamma+1)} \|\nabla u\|_2^{2(\gamma+1)} + \int_{\Gamma_1} q(x) y^2 dx + (g \diamond \nabla u)(t)}.$$

If  $\gamma(t) < \lambda_1$ , then we get

$$\begin{aligned} E(t) &\geq F(\gamma(t)) = \gamma^2(t) \left( \frac{1}{2} - \frac{1}{k} B_{\Omega}^k \gamma^{k-2}(t) \right) \\ &> \gamma^2(t) \left( \frac{1}{2} - \frac{1}{k} B_{\Omega}^k \lambda_1^{k-2} \right) \\ &> \frac{k-2}{2k} \gamma^2(t). \end{aligned} \quad (3.12)$$

**Lemma 3.2** *Let  $2 < k \leq \frac{2n-2}{n-2}$ ,  $(u_0, u_1) \in V \times L^2(\Omega)$ ,  $y_0 \in L^2(\Gamma_1)$  and hypotheses (H1)–(H2) hold. Assume further that  $E(0) < d_1$  and*

*$\gamma(0) = \sqrt{l \|\nabla u_0\|_2^2 + \frac{b}{(\gamma+1)} \|\nabla u_0\|_2^{2(\gamma+1)} + \int_{\Gamma_1} q(x) y_0^2 dx} < \lambda_1$ . Then  $E(t) < d_1$  and  $\gamma(t) = \sqrt{l \|\nabla u\|_2^2 + \frac{b}{(\gamma+1)} \|\nabla u\|_2^{2(\gamma+1)} + \int_{\Gamma_1} q(x) y^2 dx + (g \diamond \nabla u)(t)} < \lambda_1$ , for all  $t \in [0, T]$ .*

**Proof.** Using the fact that  $E(t)$  is a non-increasing function and (3.11), we get

$$E(t) \leq E(0) < d_1 \text{ and } F(\gamma(t)) < d_1 \text{ for all } t \in [0, T].$$

From (3.12) and the fact that  $E(0) < d_1$ , it follows that there exist  $\lambda_0, \lambda_2$  such that  $0 < \lambda_0 < \lambda_1 < \lambda_2$  and  $F(\lambda_0) = E(0) = F(\lambda_2)$ . As  $F(\gamma(0)) \leq E(0) = F(\lambda_0)$  and  $\lambda_0, \gamma(0) \in (0, \lambda_1)$  where the function  $F$  is increasing, then  $\gamma(0) \leq \lambda_0$ .

We argue by contradiction to prove that  $\gamma(t) \leq \lambda_0$ , for all  $t \in [0, T]$ . Suppose that there exists  $t^* \in (0, T)$  such that  $\gamma(t^*) > \lambda_0$ . We have two cases.

Case 1:  $\lambda_0 < \gamma(t^*) < \lambda_1$ , then, by virtue of the increase of  $F$  and the non-increase of  $E$ , we get

$$F(\gamma(t^*)) > F(\lambda_0) = E(0) \geq E(t^*), \text{ which contradicts (3.11).}$$

Case 2:  $\lambda_1 \leq \gamma(t^*)$ , then, by virtue of the continuity of  $\gamma$  on  $(0, t^*)$ , there exists  $t_0 \in (0, t^*)$  such that  $\lambda_0 < \gamma(t_0) < \lambda_1$ , which yields a contradiction as in case 1.

Thus,  $\gamma(t) \leq \lambda_0$ , and so  $\gamma(t) \leq \lambda_1$  for all  $t \in [0, T]$ . ■

**Theorem 3.1** *Let  $(u(t), y(t))$  be the solution of (1.1). If  $(u_0, u_1) \in V \times L^2(\Omega)$ ,  $y_0 \in L^2(\Gamma_1)$  such that,  $\gamma(0) = \sqrt{l \|\nabla u_0\|_2^2 + \frac{b}{(\gamma+1)} \|\nabla u_0\|_2^{2(\gamma+1)} + \int_{\Gamma_1} q(x) y_0^2 dx} < \lambda_1$  and  $E(0) < d_1$ , then the solution  $(u(t), y(t))$  is global in time.*

**Proof.** Using Lemma 3.2, (3.12), (3.11) and (3.1), we get

$$\begin{aligned} \frac{1}{2} \|u_t\|_2^2 + \frac{k-2}{2k} \gamma^2(t) &\leq \frac{1}{2} \|u_t\|_2^2 + F(\gamma(t)) \\ &\leq \frac{1}{2} \|u_t\|_2^2 + J(u, y)(t) = E(t) \leq E(0) < d_1, \end{aligned} \tag{3.13}$$

which ensures the boundedness of  $u$  in  $V$ ,  $u_t$  in  $L^2(\Omega)$  and  $y$  in  $L^2(\Gamma_1)$ , and hence the solution  $(u(t), y(t))$  is bounded and global in time. ■

## 4 Decay result

In order to study the decay estimate of global solution for the problem (1.1), we need the following lemma.

**Lemma 4.1** ([13]) *Let  $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a non-increasing function and  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a strictly increasing  $C^1$  function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.$$

*Assume that there exist  $\sigma \geq 0$  and  $\omega > 0$  such that*

$$\int_S^{+\infty} E^{1+\sigma}(t) \phi'(t) dt \leq \frac{1}{\omega} E^\sigma(0) E(S). \quad \text{for all } 0 \leq S < +\infty,$$

then

$$E(t) \leq E(0) \left( \frac{1 + \sigma}{1 + \omega \sigma \phi(t)} \right)^{\frac{1}{\sigma}} \quad \text{for all } t \geq 0, \quad \text{if } \sigma > 0,$$

$$E(t) \leq E(0) e^{1 - \omega \phi(t)} \quad \text{for all } t \geq 0, \quad \text{if } \sigma = 0.$$

Our main result is the following

**Theorem 4.1** *Under assumptions of Theorem 2.2 and the assumption that  $E(0) < d_1$  and  $\gamma(0) < \lambda_1$ , there exists a positive constant  $\omega$  depending on initial energy  $E(0)$ , such that the solution energy of (1.1) satisfies,*

$$E(t) \leq E(0) e^{1 - \omega \int_0^t \xi(s) ds}, \quad \text{for all } t \geq 0$$

**Remark 4.2** *If  $E(t_0) = 0$ , for some  $t_0 \geq 0$ ; then from Lemma 3.1, we have  $E(t) = 0$ , for all  $t \geq t_0$ , and then we have nothing to prove in this case. So we assume that  $E(t) > 0$ , for all  $t \geq 0$  without loss of generality.*

**Proof.** Since  $g$  is positive and  $g(0) > 0$  then for any  $t_0 > 0$  we have

$$\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0, \quad \text{for all } t \geq t_0 \quad (4.1)$$

Multiplying the first equation in (1.1) by  $\xi(t)u(t)$  and integrating by parts over  $\Omega \times (S, T)$ , with  $S \geq t_0$ , we get

$$\begin{aligned} & \int_S^T \int_{\Omega} (\xi(t)u \cdot u_t)' dx dt - \int_S^T \int_{\Omega} \xi'(t)u \cdot u_t dx dt - \int_S^T \int_{\Omega} \xi(t)u_t^2 dx dt \\ & + \int_S^T \xi(t) \left( a - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 dt + \int_S^T b\xi(t) \|\nabla u\|_2^{2(\gamma+1)} dt \\ & - \int_S^T \xi(t) \int_0^t g(t-s) \int_{\Omega} [\nabla u(s) - \nabla u(t)] \nabla u(t) dx ds dt \\ & - \int_S^T \xi(t) \int_{\Gamma_1} y_t u dx dt = \int_S^T \xi(t) \|u(t)\|_k^k dt \quad (4.2) \end{aligned}$$

since  $\gamma \geq 0$ , we deduce that

$$\begin{aligned} & \int_S^T \xi(t) \left( a - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 dt + \int_S^T \frac{b}{(\gamma+1)} \xi(t) \|\nabla u\|_2^{2(\gamma+1)} dt - \int_S^T \xi(t) \|u(t)\|_k^k dt \\ & \leq - \int_S^T \int_{\Omega} (\xi(t)u \cdot u_t)' dx dt + \int_S^T \xi'(t) \int_{\Omega} u \cdot u_t dx dt + \int_S^T \int_{\Omega} \xi(t)u_t^2 dx dt \\ & + \int_S^T \xi(t) \int_0^t g(t-s) \int_{\Omega} [\nabla u(s) - \nabla u(t)] \nabla u(t) dx ds dt + \int_S^T \xi(t) \int_{\Gamma_1} y_t u dx dt \quad (4.3) \end{aligned}$$



Recalling the definition of  $B_\Omega$  above and using (3.12), we can estimate

$$\begin{aligned}
& \frac{\|u\|_k^k}{\left( \left( a - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + \frac{b}{(\gamma+1)} \|\nabla u\|_2^{2(\gamma+1)} \right)} \\
& \leq \left( \frac{\|u(t)\|_k}{\sqrt{l \|\nabla u\|_2^2 + \frac{b}{(\gamma+1)} \|\nabla u\|_2^{2(\gamma+1)}}} \right)^k \left( \frac{2k}{k-2} E(t) \right)^{\frac{k}{2}-1} \\
& \leq B_\Omega^k \left( \frac{2k}{k-2} E(0) \right)^{\frac{k}{2}-1} \\
& < B_\Omega^k \left( \frac{2k}{k-2} \left( \frac{k-2}{2k} \right) B_\Omega^{\frac{-2k}{k-2}} \right)^{\frac{k}{2}-1} = 1
\end{aligned} \tag{4.4}$$

In view of this inequality there exists a constant  $c > 0$  independent of  $t$ , such that

$$\begin{aligned}
& c \left( \int_S^T \frac{1}{2} \xi(t) \left( a - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 dt + \int_S^T \xi(t) \frac{b}{2(\gamma+1)} \|\nabla u\|_2^{2(\gamma+1)} dt \right. \\
& \quad \left. - \frac{1}{k} \int_S^T \xi(t) \|u(t)\|_k^k dt \right) \leq \int_S^T \xi(t) \left( a - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 dt \\
& \quad + \int_S^T \frac{b}{(\gamma+1)} \xi(t) \|\nabla u\|_2^{2(\gamma+1)} dt - \int_S^T \xi(t) \|u(t)\|_k^k dt. \tag{4.5}
\end{aligned}$$

Adding some terms to both sides of the above inequality and using (4.3), we deduce

$$\begin{aligned}
& c \int_S^T \xi(t) E(t) dt \leq - \int_S^T \left( \xi(t) \int_\Omega u \cdot u_t dx \right)' dx dt + \int_S^T \xi'(t) \int_\Omega u \cdot u_t dx dt \\
& \quad + \int_S^T \xi(t) \int_0^t g(t-s) \int_\Omega [\nabla u(s) - \nabla u(t)] \nabla u(t) dx ds dt \\
& \quad + \left( 1 + \frac{c}{2} \right) \int_S^T \xi(t) \|u_t\|_2^2 dt + \int_S^T \xi(t) \int_{\Gamma_1} y_t u dx dt \\
& \quad + c \frac{1}{2} \int_S^T \xi(t) \int_{\Gamma_1} q(x) y^2 dx dt + c \frac{1}{2} \int_S^T \xi(t) (g \diamond \nabla u)(t) dt \tag{4.6}
\end{aligned}$$

Using the Cauchy-Schwarz's inequality, the Poincaré's inequalities (2.1), (2.4) and the definition of energy (3.1), we obtain estimates as follows

$$\begin{aligned}
\left| \int_{\Omega} u u_t dx \right| &\leq \sqrt{\frac{k}{l(k-2)}} C_* \left| \int_{\Omega} u_t \sqrt{\frac{l(k-2)}{k}} \frac{1}{C_*} u dx \right| \\
&\leq \frac{1}{2} \sqrt{\frac{k}{l(k-2)}} C_* \left( \int_{\Omega} u_t^2 dx + \frac{k-2}{k} \frac{l}{C_*^2} \int_{\Omega} u^2 dx \right) \\
&\leq \frac{1}{2} \sqrt{\frac{k-2}{lk}} C_* \left( \|u_t\|_2^2 + \frac{k-2}{k} l \|\nabla u\|_2^2 \right) \\
&\leq \sqrt{\frac{k-2}{lk}} C_* E(t).
\end{aligned} \tag{4.7}$$

Now, using Cauchy's inequality, (3.1), (2.4) and a technique used in [14], we get

$$\begin{aligned}
\left| \int_0^t g(t-s) \int_{\Omega} [\nabla u(s) - \nabla u(t)] \nabla u(t) dx ds \right| &\leq \epsilon \|\nabla u(t)\|_2^2 \\
&\quad + \frac{1}{4\epsilon} \int_{\Omega} \left( \int_0^t g(t-s) (\nabla u(s) - \nabla u(t)) ds \right)^2 dx \\
&\leq \epsilon \|\nabla u(t)\|_2^2 + \frac{1}{4\epsilon} \int_{\Omega} \left( \int_0^t g(t-s) ds \right) \int_0^t g(t-s) (\nabla u(s) - \nabla u(t))^2 ds dx \\
&\leq \epsilon \frac{2k}{(k-2)l} E(t) + \frac{1}{4\epsilon} (a-l) (g \diamond u)(t),
\end{aligned} \tag{4.8}$$

for any  $\epsilon > 0$ .

The following estimate is obtained using the trace theory, hypotesis (2.3), Poincaré's inequality (2.2), Hölder's and Cauchy's inequalities

$$\begin{aligned}
\left| \int_{\Gamma_1} u(t) y_t(t) dx \right| &\leq \frac{1}{\epsilon} \int_{\Gamma_1} y_t^2 dx + \epsilon \int_{\Gamma_1} u^2 dx \\
&\leq \frac{1}{\epsilon} \frac{1}{p_0} \int_{\Gamma_1} p(x) y_t^2 dx + \epsilon \overline{C}_*^2 \|\nabla u\|_2^2 \\
&\leq \frac{1}{\epsilon} \frac{1}{p_0} (-E'(t)) + \epsilon \overline{C}_*^2 \frac{2k}{(k-2)l} E(t),
\end{aligned} \tag{4.9}$$

for any  $\epsilon > 0$ ,

To estimate the sixth term of (4.6), we multiply the fourth equation in (1.1) by  $\xi(t)y$ , integrate by parts over  $(S, T) \times \Gamma_1$  and use Cauchy's inequality:

$$\begin{aligned} \int_S^T \int_{\Gamma_1} \xi(t) q(x) y^2 dx dt &\leq - \int_S^T \int_{\Gamma_1} (\xi(t) y u)' dx dt + \int_S^T \int_{\Gamma_1} \xi'(t) y u dx dt + \int_S^T \xi(t) \int_{\Gamma_1} y_t u dx dt \\ &\quad + \frac{1}{2} \int_S^T \int_{\Gamma_1} \xi(t) q(x) y^2 dx dt + \frac{1}{2} \|\xi\|_\infty \frac{p_1}{q_0} \int_S^T \int_{\Gamma_1} p(x) y_t^2 dx dt \quad (4.10) \end{aligned}$$

Using hypotesis (2.3), and (3.4), we get

$$\begin{aligned} \frac{1}{2} \int_S^T \int_{\Gamma_1} \xi(t) q(x) y^2 dx dt &\leq \left| \xi(S) \int_{\Gamma_1} y(S) u(S) \right| + \left| \xi(T) \int_{\Gamma_1} y(T) u(T) \right| + \int_S^T |\xi'(t)| \left| \int_{\Gamma_1} y u dx \right| \\ &\quad + \int_S^T \xi(t) \int_{\Gamma_1} y_t u dx dt + \frac{1}{2} \|\xi\|_\infty \frac{p_1}{q_0} \int_S^T (-E'(t)) dt \quad (4.11) \end{aligned}$$

Using the trace theory, (2.2), (3.1), Hölder's and Cauchy's inequalities, we get

$$\begin{aligned} \left| \int_{\Gamma_1} u y dx \right| &\leq \int_{\Gamma_1} \frac{1}{2q(x)} u^2 dx + \int_{\Gamma_1} \frac{q(x)}{2} y^2 dx \\ &\leq \frac{\overline{C}_*^2}{2q_0} \|\nabla u\|_2^2 + \int_{\Gamma_1} \frac{q(x)}{2} y^2 dx \\ &\leq \frac{k}{(k-2)} \left( \frac{\overline{C}_*^2}{q_0 l} + 1 \right) E(t), \end{aligned} \quad (4.12)$$

Combining now (4.9), (4.11), (4.12) and the fact that  $E$  is decreasing and  $\xi$  is bounded, we get

$$\int_S^T \int_{\Gamma_1} \xi(t) q(x) y^2 dx dt \leq c_0(\epsilon) E(S) + \epsilon c_1 \int_S^T \xi(t) E(t) dt \quad (4.13)$$

where  $c_0(\epsilon) = 4 \left( \|\xi\|_\infty \frac{k}{(k-2)} \left( \frac{\overline{C}_*^2}{q_0 l} + 1 \right) + \frac{k}{(k-2)} \left( \frac{\overline{C}_*^2}{q_0 l} + 1 \right) \|\xi\|_\infty^\theta \left\| \frac{\xi'}{\xi^\theta} \right\|_1 + \|\xi\|_\infty \frac{p_1}{q_0} + \|\xi\|_\infty \frac{1}{\epsilon} \frac{1}{p_0} \right)$  and  $c_1 = \overline{C}_*^2 \frac{4k}{(k-2)l}$ .

To estimate the third term of (4.6), we multiply the first equation in (1.1) by  $\xi(t) \int_0^t g(t-s)(u(t) - u(s)) ds$ , integrate by parts over  $\Omega \times (S, T)$  and utilize (4.1):

$$\begin{aligned}
& g_0 \int_S^T \xi(t) \|u_t\|_2^2 \leq \int_S^T \left( \xi(t) \int_\Omega \int_0^t g(t-s)(u(t)-u(s))ds \cdot u_t \right)' dx dt \\
& \quad - \int_S^T \xi'(t) \int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s))ds \right) \cdot u_t dx dt \\
& \quad - \int_S^T \xi(t) \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))ds dx dt \\
& + \int_S^T \left( \xi(t) \left( (a+b\|\nabla u\|_2^{2\gamma}) - \int_0^t g(s)ds \right) \int_0^t g(t-s) \int_\Omega \nabla u(\nabla u(t) - \nabla u(s))dx ds \right) dt \\
& \quad + \int_S^T \xi(t) \int_\Omega \left( \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds \right)^2 dx dt \\
& \quad - \int_S^T \left( \xi(t) \int_{\Gamma_1} y_t \left( \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \right) dx \right) dt \\
& \quad - \int_S^T \xi(t) \int_0^t g(t-s) \int_\Omega |u(t)|^{k-2} u(t)(u(t)-u(s))ds dx dt
\end{aligned} \tag{4.14}$$

As above, we can obtain the following estimates

$$\begin{aligned}
& \left| \int_\Omega \int_0^t g(t-s)(u(t)-u(s))u_t(t)ds dx \right| \leq \frac{1}{2} \int_\Omega u_t^2(t)dx \\
& \quad + \frac{1}{2} \int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s))ds \right)^2 dx \\
& \leq \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} (a-l) \int_0^t g(t-s) \int_\Omega (u(t)-u(s))^2 dx ds \\
& \leq \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} (a-l) C_*^2 (g \diamond u)(t) \\
& \leq \left( 1 + (a-l) C_*^2 \frac{k}{k-2} \right) E(t),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\left| \int_{\Omega} u_t(t) \int_0^t (g'(t-s)(u(t) - u(s))) ds dx \right| &\leq \epsilon \int_{\Omega} (u_t(t))^2 dx \\
&\quad + \frac{1}{4\epsilon} \int_{\Omega} \left( \int_0^t (g'(t-s)(u(t) - u(s))) ds \right)^2 dx \\
&\leq \epsilon \|u_t(t)\|_2^2 - \frac{1}{4\epsilon} g(0) \int_0^t g'(t-s) \int_{\Omega} (u(t) - u(s))^2 dx ds \\
&\leq \epsilon \|u_t(t)\|_2^2 - \frac{C_*^2}{4\epsilon} g(0) (g' \diamond u)(t) \\
&\leq \epsilon 2E(t) + \frac{C_*^2}{2\epsilon} g(0) (-E'(t)),
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
\left| \int_0^t g(t-s) \int_{\Omega} \nabla u(t) (\nabla u(t) - \nabla u(s)) dx ds \right| &\leq \int_{\Omega} \epsilon (\nabla u(t))^2 dx \\
&\quad + \frac{1}{4\epsilon} \left( \int_0^t g(s) ds \right) \left( \int_{\Omega} \int_0^t g(t-s) (\nabla u(t) - \nabla u(s))^2 dx ds \right) \\
&\leq \left( \epsilon \|\nabla u\|_2^2 + \frac{(a-l)}{4\epsilon} (g \diamond u)(t) \right),
\end{aligned} \tag{4.17}$$

Combining the estimate (4.17), the fact that  $\|\nabla u\|_2^2 \leq \frac{2k}{(k-2)l} E(t)$  and again  $\int_0^t g(s) ds \leq a-l$ , we get

$$\begin{aligned}
&\left( \left( a + b \|\nabla u\|_2^{2\gamma} \right) - \int_0^t g(s) ds \right) \int_0^t g(t-s) \int_{\Omega} \nabla u(t) (\nabla u(t) - \nabla u(s)) dx ds \\
&\leq \left( 2a - l + b \left( \frac{2k}{(k-2)l} E(0) \right)^{\gamma} \right) \left( \epsilon \frac{2k}{(k-2)l} E(t) + \frac{(a-l)}{4\epsilon} (g \diamond u)(t) \right),
\end{aligned} \tag{4.18}$$

The trace theory, hypotesis (2.3), Poincaré's inequalitie (2.2), Hölder's and Cauchy's inequalities, permit us to get

$$\begin{aligned}
& \left| \int_0^t g(t-\tau) \int_{\Gamma} y_t(t) (u(t) - u(\tau)) dx d\tau \right| \leq \frac{1}{2} \int_{\Gamma_1} y_t^2(t) dx \\
& \quad + \frac{1}{2} \int_{\Gamma_1} \left( \int_0^t g(t-\tau) (u(t) - u(\tau)) d\tau \right)^2 dx \\
& \leq \frac{1}{2} \frac{1}{p_0} \int_{\Gamma_1} p(x) y_t^2 dx + \frac{1}{2} (a-l) \int_0^t g(t-\tau) \int_{\Gamma_1} ((u(t) - u(\tau)))^2 dx d\tau \\
& \leq \frac{1}{2} \frac{1}{p_0} (-E'(t)) + \frac{1}{2} (a-l) \overline{C}_*^2 (g \diamond u)(t).
\end{aligned} \tag{4.19}$$

Using the fact that  $2 < k \leq \frac{2n-2}{n-2}$  and Poincaré's inequality (2.1), the last term on the right hand side of (4.14), can be estimated as follows

$$\begin{aligned}
& \int_0^t g(t-s) \int_{\Omega} |u(t)|^{k-2} u(t) (u(t) - u(s)) dx ds \leq \epsilon \int_{\Omega} |u(t)|^{2(k-1)} dx \\
& \quad + \frac{1}{4\epsilon} \int_{\Omega} \left( \int_0^t g(t-s) (u(t) - u(s)) ds \right)^2 dx \\
& \leq \epsilon (C_* \|\nabla u\|_2)^{2(k-1)} + \frac{1}{4\epsilon} (a-l) C_*^2 (g \diamond u)(t) \\
& \leq \epsilon C_*^{2(k-1)} \left( \frac{2k}{(k-2)l} E(t) \right)^{k-1} + \frac{1}{4\epsilon} (a-l) C_*^2 (g \diamond u)(t) \\
& \leq \epsilon \left( \frac{2k C_*^2}{(k-2)l} \right)^{k-1} (E(0))^{k-2} E(t) \\
& \quad + \frac{1}{4\epsilon} (a-l) C_*^2 (g \diamond u)(t).
\end{aligned} \tag{4.20}$$

Combining estimates (4.14)-(4.20), we get

$$\begin{aligned}
& \int_S^T \xi(t) \|u_t\|_2^2 \leq c_2(\epsilon) E(S) + \epsilon c_3 \int_S^T \xi(t) E(t) dt + c_4(\epsilon) \int_S^T \xi(t) (g \diamond u)(t) dt \\
& \quad \text{where } c_2(\epsilon) = g_0^{-1} \left( \left( 2\xi_{\infty} + \|\xi\|_{\infty}^{\theta} \left\| \frac{\xi'}{\xi^{\theta}} \right\|_1 \right) \left( 1 + (a-l) C_*^2 \frac{k}{k-2} \right) + \frac{\xi_{\infty}}{2} \left( \frac{C_*^2}{\epsilon} g(0) + \frac{1}{p_0} \right) \right), \\
& \quad c_3 = g_0^{-1} \left( 2 + \frac{2k(2a-l+b(\frac{2k}{(k-2)l} E(0))^{\gamma})}{(k-2)l} + \left( \frac{2k C_*^2}{(k-2)l} \right)^{k-1} (E(0))^{k-2} \right) \text{ and} \\
& \quad c_4(\epsilon) = g_0^{-1} \left( \frac{(2a-l+b(\frac{2k}{(k-2)l} E(0))^{\gamma})(a-l)}{4\epsilon} + (a-l) \left( 1 + \frac{1}{2} \overline{C}_*^2 + \frac{1}{4\epsilon} C_*^2 \right) \right).
\end{aligned} \tag{4.21}$$

From hypothesis (2.5) and (2.6), it follows that

$$\begin{aligned}
\int_S^T \xi(t) (g \diamond \nabla u)(t) dt &\leq e^r \int_S^T \int_0^t \xi(t-s) g(t-s) \|\nabla u(t) - \nabla u(s)\|_2^2 ds dt \\
&\leq e^r \int_S^T -(g' \diamond \nabla u)(t) dt \\
&\leq e^r \int_S^T (-2E'(t)) dt \\
&\leq 2e^r E(S)
\end{aligned} \tag{4.22}$$

Consequently, using (4.6), (4.7), (4.8), (4.21), (4.9), (4.13), (4.22), (2.6) and the fact that  $E$  is decreasing, we get

$$c \int_S^T \xi(t) E(t) dt \leq c_5(\epsilon) E(S) + \epsilon c_6 \int_S^T \xi(t) E(t) dt \tag{4.23}$$

where  $c_5(\epsilon) = \sqrt{\frac{k-2}{lk}} \left( 2 \|\xi\|_\infty C_* + C_* \|\xi\|_\infty^\theta \left\| \frac{\xi'}{\xi^\theta} \right\|_1 \right) + c_2(\epsilon) \left( 1 + \frac{\epsilon}{2} \right) + \left( \frac{1}{2\epsilon} (a-l) + 2c_4(\epsilon) (1+c) \right) e^r$   
 $+ \frac{1}{\epsilon} \frac{1}{p_0} \|\xi\|_\infty + \frac{\epsilon}{2} c_0(\epsilon)$  and  $c_6 = \frac{2k}{(k-2)l} \left( 1 + \overline{C}_*^2 \right) + c_3 + (c_1 + c_3) \frac{\epsilon}{2}$   
and so, for  $\epsilon$  small enough, we get

$$\int_S^T \xi(t) E(t) dt \leq \frac{c_5(\epsilon)}{(c - \epsilon c_6)} E(S), \text{ for all } S \geq t_0, \tag{4.24}$$

and if  $0 \leq S < t_0$ , it suffices to observe that

$$\begin{aligned}
\int_S^T \xi(t) E(t) dt &= \int_S^{t_0} \xi(t) E(t) dt + \int_{t_0}^T \xi(t) E(t) dt \\
&\leq E(S) \int_0^{t_0} \xi(t) dt + \frac{c_5(\epsilon)}{(c - \epsilon c_6)} E(t_0)
\end{aligned} \tag{4.25}$$

therefore  $\int_S^T \xi(t) E(t) dt \leq C E(S)$ , for all  $S \geq 0$  and some constant  $C \geq 0$  independent of  $S$  and  $T$

Let  $T \rightarrow +\infty$ , applying Lemma 4.1 (with  $\sigma = 0$  and  $\phi(t) = \int_0^t \xi(\tau) d\tau$ ), we conclude that  $E(t) \leq E(0) e^{1-\omega \int_0^t \xi(\tau) d\tau}$ , for all  $t \geq 0$ , for some  $\omega = \omega(E(0), \xi; t_0)$

■

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